Published by Institute of Physics Publishing for SISSA/ISAS

Received: *October 25, 2004*
Revised: *February 15, 2005*
Accepted: *February 18, 2005*

Explicit formulas for the scalar modes in Seiberg-Witten theory with an application to the Argyres-Douglas point

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Abstract: General formulas for the scalar modes a_i and a_D^i in the Seiberg-Witten SU(N) setting are derived in the cases with and without massive hypermultiplets. Subsequently these formulas are applied in a study of the SU(3) Argyres-Douglas point. The paper collects, in an appendix, various facts on the function Lauricella $F_D^{(n)}$, which naturally appears in the derived formulas.

Keywords: Duality in Gauge Field Theories, Supersymmetry and Duality, Supersymmetric Effective Theories.

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http://jhep.sissa.it/archive/papers/jhep022005057/jhep022005057.pdf

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1. Introduction

In a much celebrated work [1], Seiberg and Witten found an exact solution to $N = 2$ supersymmetric four-dimensional Yang-Mills theory with gauge group SU(2). This paper initiated a whole new, tree-sized, branch of research leading to a vast set of exactly solvable Yang-Mills theories in various dimensions and with various degrees of supersymmetry. For some basic or introductory works see, for example, [2]–[8], and references therein. Of particular interest for these solutions is the understanding of the moduli space of vacua, which in many cases turns out to be a hyperelliptic Riemann surface. In particular, simply-laced Lie groups lead to spectral curves which are hyperelliptic.

The BPS spectrum of such a model is entirely determined by the periods of a special meromorphic differential 1-form on this Riemann surface, the Seiberg-Witten differential λ_{SW}. A general hyperelliptic Riemann surface Σ can be described in terms of two variables w, Z in the polynomial form

$$w^2 + 2A(Z)w + B(Z) = 0 \tag{1.1}$$

with $A(Z), B(Z) \in \mathbb{C}[Z]$. After a simple coordinate transformation in $y = w - A(Z)$, this takes on the more familiar form $y^2 = A(Z)^2 - B(Z)$. But we might also write the hyperelliptic curve in terms of a rational map if we divide the defining equation 1.1 by $A(Z)^2$ and put $\tilde{w} = w/A(Z) + 1$ to arrive at the representation

$$(1 - \tilde{w})(1 + \tilde{w}) = \frac{B(Z)}{A(Z)^2} \,. \tag{1.2}$$

This form is very appropriate in the frame of Seiberg-Witten models, since the Seiberg-Witten differential can be read off directly: The rational map $R(Z) = B(Z)/A(Z)^2$ is singular at the zeroes of $B(Z)$ and $A(Z)$, and is degenerate whenever its wronskian $W(R) \equiv W(A(Z)^2, B(Z)) = (\partial_Z A(Z)^2)B(Z) - A(Z)^2(\partial_Z B(Z))$ vanishes. This is precisely the information encoded in λ_{SW} which for arbitrary hyperelliptic curves, given by a rational map $R(Z) = B(Z)/A(Z)^2$, can be expressed as

$$\lambda_{\mathrm{SW}} = \frac{Z}{2\pi\mathrm{i}}\,\mathrm{d}\left(\log\frac{1-\tilde{w}}{1+\tilde{w}}\right) = \frac{1}{2\pi\mathrm{i}}\mathrm{d}(\log R(Z))\frac{Z}{\tilde{w}} = \frac{1}{2\pi\mathrm{i}}\frac{W(A(Z)^2, B(Z))}{A(Z)B(Z)}\frac{Z\,\mathrm{d}Z}{y} \,. \tag{1.3}$$

Note that the denominator polynomial being a square guarantees the curve to be hyperelliptic.

The crucial point in Seiberg-Witten theory is that the physically relevant information of the BPS mass spectrum is encoded in the periods of the above defined 1-form. In general, the hyperelliptic curve has $2N_c$ branch points and thus is of genus $g = N_c - 1$. Any element γ of the homology of the curve defines a period and hence the mass of a BPS state,

$$Z_\gamma = \oint_\gamma \lambda_{\mathrm{SW}} \,, \qquad m_\gamma = |Z_\gamma| \,. \tag{1.4}$$

If N_f (massive) hypermultiplets are present, the homology basis gets extended by N_f loops encircling each of the differential's poles in order to pick up residue terms, if the contour γ is deformed in such a way that it crosses a pole.

The appealing picture emerging from the analysis of the Seiberg-Witten solution of supersymmetric low-energy effective field theories is that the geometry of an auxiliary Riemann surface Σ entirely determines the physics. Given the surface and a special 1-form on it, the properties of particles are encoded in the periods of this particular 1-form, the Seiberg-Witten differential. Choosing a canonical homology basis $\{\alpha_i, \beta^i\}$ with $1 \leq i < N_c$, such that $\alpha_i \cap \alpha_j = \beta^i \cap \beta^j = 0$ and $\alpha_i \cap \beta^j = \delta_i^{\,j}$, defines electric-like and magnetic-like quantum numbers for particles. More precisely, every $\gamma \in H_1(\Sigma, \mathbb{Z})$ can be written as a linear combination with integer coefficients,

$$\gamma = \sum_i (q^i \alpha_i + g_i \beta^i) \,, \tag{1.5}$$

defining in turn dual pairs of quantum numbers (q^i, g_i). Stable particles are those, for which all these pairs consist of coprime numbers. As we see, the definition of 'electric' and 'magnetic' quantum numbers is fixed by the choice of a basis in the homology of Σ. It is generically independent of small variations of the location of the branch points of the

Riemann surface, i.e. the basis is generically invariant under small variations of the moduli of the surface. However, there is typically no single canonical homology basis, which can be used for all possible variations of the moduli. This will concern us in some detail later on. This is an important point, since varying the moduli usually implies a deformation of the contours, which encircle, once and for all, chosen pairs of branch points. However, since the Seiberg-Witten differential has zeroes and poles, such deformations are not always possible without crossing one of these.

The approach taken in this paper emphasizes the geometrical picture for the BPS states, namely that they are described in terms of period integrals taken along homology contours. In this picture, one immediately sees which BPS states can become massless under what circumstances. This is especially easy to see, when the Riemann surface is realized as a branched covering of the complex plane (or sphere). As soon as two branch points coalesce, the BPS state corresponding to the cycle γ_{ij}, which encloses precisely the two branch points e_i and e_j and none of the poles of the Seiberg-Witten differential, will become massless, since $Z_{\gamma_{ij}} \to 0$ for $\gamma_{ij} \to 0$, i.e. for $|e_i - e_j| \to 0$ under a suitable variation of the moduli. Another important advantage of our approach is that we do not need to find Picard-Fuchs equations, which is a very hard task for gauge groups of rank greater than two.

1.1 Overview of results and structure of the paper

Working with a completely factorized form of the Seiberg-Witten differential we derive general formulas for the scalar modes in Seiberg-Witten theory for gauge group SU(N) in the cases with and without massive hypermultiplets which involve the multiple hypergeometric function Lauricella $F_D^{(n)}$. These formulas work directly in the picture of a branched ramified covering of the complex plane and thus do not rely on the derivation and solutions of the associated system of Picard-Fuchs equations.

As a test case for our formulas we consider the semiclassical asymptotics of the scalar modes for the case of gauge group SU(2), showing in detail how the formulas reproduce the standard result known in the literature.

Then, as a more substantial application, we consider Argyres-Douglas' $\mathbb{Z}_3$-point.

The paper closes with a rather long appendix on the Lauricella functions $F_D^{(n)}$ which naturally appear in our derived formulas. The idea is to collect all necessary results on these functions in order to make the paper more self-contained.

2. General formulas for the scalar modes

In this section we derive our formulas for the scalar modes occuring in the BPS mass formula. As was remarked in the introduction, one of our devices is the factorization of the Seiberg-Witten differential so that the integrals involved in our calculations can be identified as Lauricella functions of type D.

In the first subsection we gather the relevant expression for the Seiberg-Witten differential and so on. Then, in the second subsection, we consider the SU(N) case without massive hypermultiplets, whereas in the third subsection we consider the case with massive hypermultiplets.

For completeness, we mention that integration of the Seiberg-Witten form can also be achieved by other methods. First of all, the traditional way is by setting up and solving the Picard-Fuchs system of the curve Σ with given form λ_{SW}. Another method integrates the Seiberg-Witten differential directly, after recasting it in modular coordinates with respect to the charge lattice generated by its periods [9]. Conformal invariance or isomonodromic properties of the periods can also be used to find them in explicit form [10, 11]. Finally, for certain cases such as SU(2), functional properties of the prepotential related to a certain kind of modular invariance can also be employed [12, 13], see also [14] for an approach where the prepotential is computed directly, without determining the scalar modes first. Most of these methods become increasingly more difficult or cannot be applied at all for higher rank gauge groups.

2.1 The Seiberg-Witten differential

As is well known, the exact Seiberg-Witten solutions of SUSY Yang-Mills theories are obtained like this: One writes down a certain hyperelliptic curve,[1] often called the spectral curve of the theory, and the BPS spectrum for the particular theory is then determined by the periods of a given meromorphic 1-form λ_{SW} on this curve.

For $N=2$ supersymmetric SU(N) Yang-Mills theory without massive hypermultiplets the curve-defining equation is (the N is the N in SU(N))

$$y^2 = A(x)^2 - B := \left(x^N - \sum_{k=2}^{N} u_k x^{N-k}\right)^2 - \Lambda^{2N} = \prod_{i=1}^{2N}(x-e_i)\,, \tag{2.1}$$

and the Seiberg-Witten differential is

$$\lambda_{\mathrm{SW}} = \frac{1}{2\pi\mathrm{i}} \frac{\prod_{\ell=0}^{N-1}(x-z_\ell)}{\prod_{i=1}^{2N}\sqrt{x-e_i}}\,\mathrm{d}x\,, \tag{2.2}$$

where the $z_\ell\,(\ell>0)$ are the zeros of $2A'(x)B$, $z_0=0$ and the e's are the zeros of y^2. Usually the e's are called *branch points*.

This form of the Seiberg-Witten differential is not unique. We could add exact terms to it which would not affect the computation of period integrals.

With these data one proceeds as follows: Let $\{\alpha_i,\beta^i\}_{1\le i\le N-1}$ denote a canonical homology basis for our curve, i.e. one for which $\alpha_i\cap\beta^j = \delta_i{}^j$, then the scalar modes are given by $a_i = \int_{\alpha_i}\lambda_{\mathrm{SW}}$, $a_D^j = \int_{\beta^j}\lambda_{\mathrm{SW}}$ and in terms of these quantities the mass of a BPS state with charge $(\mathbf{q},\mathbf{g})$ is given by $m_{(\mathbf{q},\mathbf{g})} \propto |q^i a_i + g_j a_D^j|$. In a string-theoretic derivation, the homology cycles arise as projections from three-cycles in Calabi-Yau compactifications.

The case with massive hypermultiplets brings about some changes in the spectral curve as well as in λ_{SW} which we shall communicate in due course (see section 2.3 below).

[1]To be precise, there are cases where the curve is not hyperelliptic. This happens for gauge groups which are not simply-laced. However, we shall not consider those cases.

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In deriving our results, we will be making use of the following integral identity involving Lauricella $F_D^{(n)}$; this function is considered more fully in the appendix as well as in [15]:

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}\prod_{i=1}^{n}(1-tx_i)^{-b_i}\,\mathrm{d}t = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\,F_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)\,. \tag{2.3}$$

Briefly, $F_D^{(n)}$ is a function of several (namely n) complex variables, defined by the power series

$$\begin{aligned} &F_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n) = \\ &\quad = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}m_1!\cdots m_n!}\,x_1^{m_1}\cdots x_n^{m_n}\,, \end{aligned} \tag{2.4}$$

whenever $|x_1|,\ldots,|x_n|<1$ and by analytic continuation elsewhere. As a matter of notation, $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ denotes the Pochhamer symbol (rising factorial).

2.2 SU(N) without hypermultiplets

Using the notation from the previous section, let us choose a canonical homology basis $B=\{\alpha_i,\beta^i\}_i$ for our curve, such that for any $\gamma\in B$ we have

$$\int_\gamma \lambda_{\mathrm{SW}} = 2\int_{e_i}^{e_j}\lambda_{\mathrm{SW}}\,, \tag{2.5}$$

for some branch points e_i, e_j.

Geometrically speaking, the existence of such a basis means that if γ encircles e_i and e_j, no other e lies on the straight line connecting those two points, so that the original contour integral can be converted into twice the integral along that line.

However, for some configurations of the e's no such basis might exist and therefore, unless stated otherwise, we shall explicitly *assume* its existence, treating it as an *hypothesis* for what follows.

As stated above, the quantity a_i (a_D^i) is obtained by integrating the Seiberg-Witten differential λ_{SW} along the cycle α_i (β^i) for which we indiscriminately write γ.

Thus we have

$$\begin{aligned} a_i \quad \text{or} \quad a_D^i &= \int_\gamma \lambda_{\mathrm{SW}} \\ &= 2\int_{e_i}^{e_j}\lambda_{\mathrm{SW}} \\ &= \frac{1}{\pi\mathrm{i}}\int_{e_i}^{e_j}\prod_{k=0}^{N-1}(x-z_k)\prod_{\ell=1}^{2N}(x-e_\ell)^{-\frac{1}{2}}\,\mathrm{d}x \\ &= \frac{1}{\pi\mathrm{i}}(e_j-e_i)\int_0^1\prod_{k=0}^{N-1}\big(e_i-(e_i-e_j)t-z_k\big)\prod_{\ell=1}^{2N}\big(e_i-(e_i-e_j)t-e_\ell\big)^{-\frac{1}{2}}\,\mathrm{d}t \end{aligned}$$

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$$= \frac{1}{\pi \mathrm{i}}(e_j - e_i) \prod_{k=0}^{N-1} (e_i - z_k)(e_i - e_j)^{-\frac{1}{2}}(e_j - e_i)^{-\frac{1}{2}} \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{2N} (e_i - e_j)^{-\frac{1}{2}} \times$$

$$\times \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}} \prod_{k=0}^{N-1} \left(1 - t\frac{e_i - e_j}{e_i - z_\ell}\right) \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{2N} \left(1 - t\frac{e_i - e_j}{e_i - e_\ell}\right)^{-\frac{1}{2}} \mathrm{d}t$$

$$= (e_i - e_j)^{\frac{1}{2}} \prod_{k=0}^{N-1} (e_i - z_k) \prod_{\substack{\ell=1 \\ \ell \neq i,j}}^{2N} (e_i - e_\ell)^{-\frac{1}{2}} \times \tag{2.6}$$

$$\times F_D^{(3N-2)}\left(\frac{1}{2}, \underbrace{-1, \ldots, -1}_{N\,\text{parameters}}, \underbrace{\frac{1}{2}, \ldots, \frac{1}{2}}_{\substack{2N-2 \\ \text{parameters}}}; 1; \underbrace{\left\{\frac{e_i - e_j}{e_i - z_k}\right\}_k}_{N\,\text{variables}}, \underbrace{\left\{\frac{e_i - e_j}{e_i - e_\ell}\right\}_{\ell \neq i,j}}_{2N-2\,\text{variables}}\right),$$

where in the last line we have made use of the integral identity 2.3.

This is the main result of this section.

By a slight change in notation in this formula we can greatly shift emphasis towards a more 'geometric' viewpoint.

To this end we make the following definitions

$$\zeta := e_i - e_j, \tag{2.7}$$

$$\xi_\nu := \frac{e_i - e_j}{e_i - z_{\nu-1}}, \qquad \nu \in \{1, \ldots, N\}; \tag{2.8}$$

$$x_\nu := \frac{e_i - e_j}{e_i - e_\nu}, \qquad \nu \in \{1, \ldots, i-1, i+1, \ldots, 2N\}. \tag{2.9}$$

Note that x_i is *not defined.*

Furthermore, let

$$b_\xi := \underbrace{(-1, \ldots, -1)}_{N\,\text{components}}, \tag{2.10}$$

$$b_x := \underbrace{\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)}_{2N-2\,\text{components}}. \tag{2.11}$$

Then the result 2.6 reads

$$a_i \quad \text{or} \quad a_D^i = \zeta \prod_{\nu=1}^{N} \xi_\nu^{-1} \prod_{\substack{\nu=1 \\ \nu \neq i}}^{2N} x_\nu^{\frac{1}{2}} F_D^{(3N-2)}\left(\frac{1}{2}, b_\xi, b_x; 1; \{\xi_\nu\}_{\nu=1}^N, x_1, \ldots, \widehat{x_j}, \ldots, x_{2N}\right), \tag{2.12}$$

where $\widehat{\ }$ denotes omission. We remark that in total there are $3N-2$ 'x'-arguments to the F_D, x_i being undefined and x_j being omitted.

The promised shift in emphasis takes place if we now consider a_i and a_D^i as functions solely of ζ and the ξ's and x's, the domain of these 'new' functions being the subset of

$\mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^{2N-2}$ which results if we regard ζ and the ξ's and x's as functions of the u's — the vacuum expectation values — and let the latter vary freely over the whole of $\mathbb{C}^{N-1}$.

Unfortunately it turns out that the described domain is rather complicated. Nevertheless, we believe that this viewpoint sheds some additional light on the structure of the moduli space.

The Lauricella function of type F_D has been used in the context of Seiberg-Witten theory before [11, 16]. However, the approach taken there makes use of conformal invariance of the Seiberg-Witten periods which leads to a different set of variables. The main problem with this older approach is that the variables involve crossing ratios of the branch points and zeroes of the Seiberg-Witten differential. This makes an analysis of singular points, as we will perform it in simple examples below, almost impossible.

2.3 SU(N) with hypermultiplets

As remarked in section 2.1 the case where massive hypermultiplets are present brings about some modifications in the Seiberg-Witten differential and the curve on which it is defined. Namely, if we denote the gauge group by SU(N_c) and if there are present N_f hypermultiplets with masses m_r, the curve-defining equation becomes

$$y^2 = A(x)^2 - B(x) := \left(x^{N_c} - \sum_{k=2}^{N_c} u_k x^{N_c-k} \right)^2 - \Lambda^{2N_c-N_f} \prod_{r=1}^{N_f}(x-m_r) = \prod_{i=1}^{2N_c}(x-e_i)\,, \quad (2.13)$$

and the Seiberg-Witten differential then reads

$$\lambda_{\mathrm{SW}} = \frac{1}{2\pi \mathrm{i}} \frac{\prod_{\ell=0}^{N_c+N_f-1}(x-z_\ell)}{\prod_{i=1}^{2N}\sqrt{x-e_i}\prod_{j=1}^{N_f}(x-m_j)}\,\mathrm{d}x\,, \quad (2.14)$$

where the $z_\ell\,(\ell > 0)$ now denote the zeros of $2A'(x)B(x) - A(x)B'(x)$, and again $z_0 = 0$.

Under the same conditions[2] on our homology basis as in the previous section we obtain by a calculation absolutely analogous to that of 2.6

$$a_i \quad \text{or} \quad a_D^i = (e_i - e_j)^{\frac{1}{2}} \prod_{k=0}^{N_c+N_f-1}(e_i - z_k) \prod_{\substack{\ell=1\\ \ell\neq i,j}}^{2N_c}(e_i-e_\ell)^{-\frac{1}{2}} \prod_{r=1}^{N_f}(e_i-m_r)^{-1} \times \quad (2.15)$$

$$\times F_D^{(3N_c+2N_f-2)}\Bigg(\frac{1}{2}, \underbrace{-1,\ldots,-1}_{N_c+N_f\text{ parameters}}, \underbrace{\frac{1}{2},\ldots,\frac{1}{2}}_{2N_c-2\text{ parameters}}, \underbrace{1,\ldots,1}_{N_f\text{ parameters}};1;$$

$$\underbrace{\left\{\frac{e_i-e_j}{e_i-z_k}\right\}_k}_{N_c+N_f\text{ variables}}, \underbrace{\left\{\frac{e_i-e_j}{e_i-e_\ell}\right\}_{\ell\neq i,j}}_{2N_c-2\text{ variables}}, \underbrace{\left\{\frac{e_i-e_j}{e_i-m_r}\right\}_r}_{N_f\text{ variables}}\Bigg)\,. \quad (2.16)$$

This is the result for the case with massive hypermultiplets.

[2] This is not entirely true, because now we must also make sure, that no m_r lies on the straight line connecting e_i and e_j and we also have to enlarge the homology basis to include cycles round the poles of the Seiberg-Witten differential as explained in the introduction. The corresponding scalar modes are suppressed here, since their calculation reduces to the evaluation of residues.

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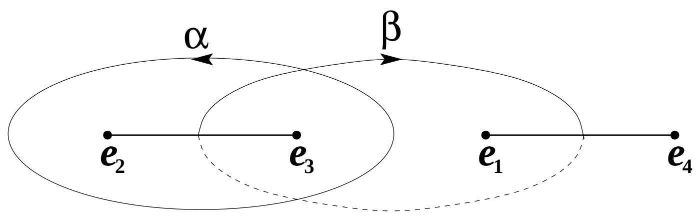

Figure 1: Homology basis for SU(2).

Of course one could also clean up the notation in this formula, but since we will not be making any use of it we shall pass this by.

3. Applications

In this section we discuss applications of the general formulas developed earlier. First, as a warm up, we show how some well known results (namely the asymptotics for the scalar modes $a(u)$ and $a_D(u)$ as $u \to \infty$) for the SU(2) case can be obtained quite easily. After that, we study the Argyres-Douglas $\mathbb{Z}_3$-point in the SU(3) case.

3.1 SU(2)

Let us calculate the scalar modes when the gauge group is SU(2). Referring to the curve 2.1, here $N = 2$ and there is exactly one modulus u.

The z's (cf. the line below eq. 2.2) are

$$z_0 = 0\,, \qquad z_1 = 0 \tag{3.1}$$

and the branch points are

$$\begin{aligned} e_1 &= \sqrt{u-\Lambda^2}\,, &(3.2)\\ e_2 &= -\sqrt{u+\Lambda^2}\,, &(3.3)\\ e_3 &= -\sqrt{u-\Lambda^2}\,, &(3.4)\\ e_4 &= \sqrt{u+\Lambda^2}\,. &(3.5)\end{aligned}$$

Taking this into account, we have the situation depicted in figure 1.

Therefore, using 2.6 (our homology basis $\{\alpha, \beta\}$ is shown in figure 1),

$$\begin{aligned} a(u) &= \int_\alpha \lambda_{\rm SW} \\ &= -2\int_{e_2}^{e_3} \lambda_{\rm SW} \\ &= -\frac{e_2^2}{(e_2-e_1)^{\frac{1}{2}}(e_2-e_4)^{\frac{1}{2}}}\, F_D^{(4)}\left(\ldots; \frac{e_2-e_3}{e_2-0}, \frac{e_2-e_3}{e_2-0}, \frac{e_2-e_3}{e_2-e_1}, \frac{e_2-e_3}{e_2-e_4}\right) \\ &\sim \frac{1}{\sqrt{8}}\sqrt{2u}\,, \quad u \to \infty\,, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
a_D(u) &= \int_\beta \lambda_{\text{SW}} \\
&= -2\int_{e_3}^{e_1} \lambda_{\text{SW}} \\
&= -\frac{e_3^2}{(e_3-e_2)^{\frac{1}{2}}(e_3-e_4)^{\frac{1}{2}}} F_D^{(4)}\left(\ldots; \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\right) \\
&\sim -\frac{1}{\pi}\frac{e_3^2}{(e_1-e_3)^{\frac{1}{2}}(e_3-e_4)^{\frac{1}{2}}} \log\left(\frac{e_1-e_3}{e_3-e_2}\right), \quad u\to\infty \\
&\sim \frac{1}{\sqrt{8}}\frac{\mathrm{i}}{\pi}\sqrt{2u}\,\log u, \quad u\to\infty\,,
\end{aligned} \tag{3.7}$$

where for $a_D(u)$ we have made use of equation A.11 in the appendix.

These results are of course well known.

The rigorous minded reader might want to consider the following paragraph, where we explain in a little more detail how we applied A.11 to obtain the asymptotic behavior of $a_D(u)$.

Details of previous derivation: for the following it is useful to write out in full the F_D in formula 3.7:

$$F_D^{(4)}\left(\frac{1}{2}, -1, -1, \frac{1}{2}, \frac{1}{2}; 1; \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-0}, \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\right). \tag{3.8}$$

Since the first two arguments are identical and equal to 2 this reduces to

$$F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; 2, \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\right), \tag{3.9}$$

as is easily seen from the power series representation 2.4. Using the power series even though $\frac{e_3-e_1}{e_3-0} \equiv 2$ is legal, since the corresponding parameters are negative integers so that this part of the series terminates after a finite number of terms, giving something polynomial in the argument.

Following up on the remark 'something polynomial' we obtain

$$\begin{aligned}
F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; \mathbf{2}, \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\right) = {}& F_D^{(2)}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\right) - \\
& -\mathbf{2}F_D^{(2)}\left(\frac{1}{2}{+}1, \frac{1}{2}, \frac{1}{2}; 1{+}1; \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\right) + \\
& +\frac{3}{8}\mathbf{2}^2 F_D^{(2)}\left(\frac{1}{2}{+}2, \frac{1}{2}, \frac{1}{2}; 1{+}2; \frac{e_3-e_1}{e_3-e_2}, \frac{e_3-e_1}{e_3-e_4}\right).
\end{aligned} \tag{3.10}$$

Now, in the limit $u\to\infty$, $|\frac{e_3-e_1}{e_3-e_2}| \to \infty$ and $|\frac{e_3-e_1}{e_3-e_4}| \nearrow 1$, the divergence being faster than the convergence. Therefore we can employ the analytic continuation formula A.11. Doing so, one easily verifies that the first asymptotic in 3.7 is correct. The expression in the fourth line is asymptotically equal to the one before and since the relation $\sim$ is transitive everything is in perfect order.

3.2 Argyres-Douglas' $\mathbb{Z}_3$-point

In this section we apply our main result 2.6 to the $\mathbb{Z}_3$-point discovered by Argyres and Douglas (cf. [17, 18]). Also, [19, 20, 21], and references cited therein, have made contributions to this subject. The case with hypermultiplets was considered in [22, 23].

Roughly speaking, the Argyres-Douglas point is interesting because it provides us with an example of a theory in which the BPS spectrum contains a pair of dual particles, i.e. particles where one is electrically and the other magnetically charged, which becomes simultaneously massless. Furthermore, at such points in moduli space, the theory becomes superconformal which, of course, has important consequences [24].

3.2.1 Vanishing of scalar modes

The $\mathbb{Z}_3$-point is the e-configuration in the SU(3) case determined by the choice of vev's $u = 0$, $v = \Lambda^3$ (for completeness, we mention that there exist other choices which also lead to '$\mathbb{Z}_3$-points'). Our aim is to apply our formula 2.6 to this case.

Therefore, following Argyres and Douglas, we write

$$u = \delta u\,, \qquad v = \Lambda^3 + \delta v\,. \tag{3.11}$$

We shall restrict ourselves to (real) $\delta u < 0$ and (real) $\delta v > 0$. We remark on these hypotheses in section 3.2.2 below.

Referring to the curve 2.1, the e's (branch points) are the zeros of

$$\left(x^3 - \delta u\, x - (\Lambda^3 + \delta v)\right)^2 - \Lambda^6\,. \tag{3.12}$$

They are easily seen to be separated into two classes: Those which are the zeros of $p_1(x) = x^3 - \delta u\, x - \delta v$ and those which are the zeros of $p_2(x) = x^3 - \delta u\, x - \delta v - 2\Lambda^3$.

Put $P = \mathrm{sgn}(-\delta v/2)\,\sqrt{|-\delta u/3|}$ and $\beta = \frac{1}{3}\mathrm{arsinh}\frac{-\delta v/2}{P^3}$.

Then, under our assumptions on the δ's, the zeros of p_1 are (see e.g. [25])

$$\begin{aligned} e_1 &= -2P\sinh\beta\,,\\ e_2 &= P(\sinh\beta - \mathrm{i}\sqrt{3}\cosh\beta)\,,\\ e_3 &= P(\sinh\beta + \mathrm{i}\sqrt{3}\cosh\beta)\,. \end{aligned} \tag{3.13}$$

We shall not need explicit formulas for the zeros of p_2. It will suffice to know that whenever the δ's are small, the zeros of p_2 (call them e_4,e_5,e_6) will be near the third roots of $2\Lambda^3$.

The resulting configuration, along with our choice of homology basis, is visualized in figure 2. Before writing down our expressions for the a's and a_D's we also need to know that the z's (recall the definition just below eq. 2.2) now are

$$z_0 = 0\,, \qquad z_1 = \sqrt{\frac{\delta u}{3}}\,, \qquad z_2 = -\sqrt{\frac{\delta u}{3}}\,. \tag{3.14}$$

In the study of the $\mathbb{Z}_3$-point one is mostly interested in a_1 and a_D^1 because it are these two dual quantities which simultaneously vanish at the $\mathbb{Z}_3$-point. This, of course, is related

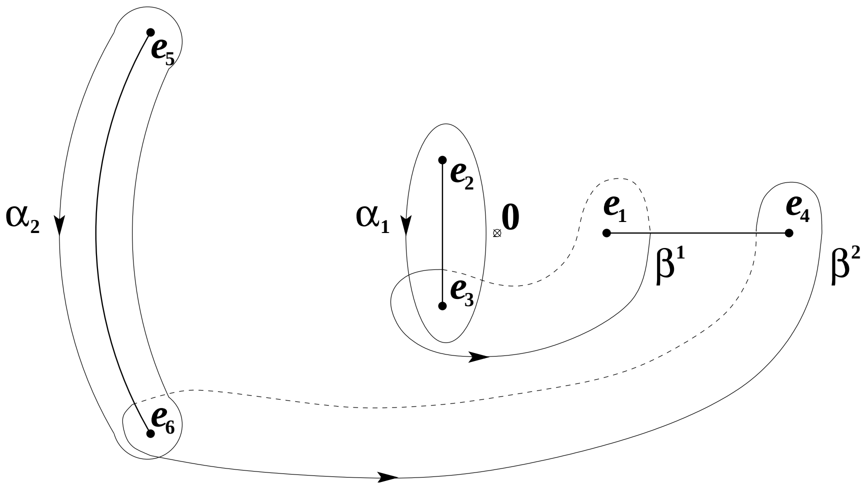

Figure 2: Homology basis for SU(3).

to the fact that as the δ's tend to 0 the cycles α_1 and β^1 contract to points because e_1,e_2,e_3 tend to 0.

As functions of δu and δv, i. e. near the Argyres-Douglas point, the scalar modes a_1, a_D^1 are given by (cf. eq. 2.6)

$$\begin{aligned} a_1 &= 2\int_{e_2}^{e_3} \lambda_{\rm SW} \\ &= \frac{e_2(e_2^2-\delta u/3)}{(e_2-e_1)^{\frac{1}{2}}(e_2-e_4)^{\frac{1}{2}}(e_2-e_5)^{\frac{1}{2}}(e_2-e_6)^{\frac{1}{2}}}\times \\ &\quad \times F_D^{(7)}\left(\ldots;\frac{e_2-e_3}{e_2-0},\frac{e_2-e_3}{e_2-\sqrt{\delta u/3}},\frac{e_2-e_3}{e_2+\sqrt{\delta u/3}},\frac{e_2-e_3}{e_2-e_1},\frac{e_2-e_3}{e_2-e_4},\frac{e_2-e_3}{e_2-e_5},\frac{e_2-e_3}{e_2-e_6}\right), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} a_D^1 &= 2\int_{e_3}^{e_1} \lambda_{\rm SW} \\ &= \frac{e_3(e_3^2-\delta u/3)}{(e_3-e_2)^{\frac{1}{2}}(e_3-e_4)^{\frac{1}{2}}(e_3-e_5)^{\frac{1}{2}}(e_3-e_6)^{\frac{1}{2}}}\times \\ &\quad \times F_D^{(7)}\left(\ldots;\frac{e_3-e_1}{e_3-0},\frac{e_3-e_1}{e_3-\sqrt{\delta u/3}},\frac{e_3-e_1}{e_3+\sqrt{\delta u/3}},\frac{e_3-e_1}{e_3-e_2},\frac{e_3-e_1}{e_3-e_4},\frac{e_3-e_1}{e_3-e_5},\frac{e_3-e_1}{e_3-e_6}\right). \end{aligned} \tag{3.16}$$

The next thing we shall do is to examine the behavior of a_1 and a_D^1 as the δ's both tend to 0. There are several meanings one can attach to the phrase 'tend to 0.' One possible meaning is the usual one from the calculus of several (complex) variables, where both variables are considered as *independent* of each other. Another possible meaning, and this is often done in physics, is to consider both variables as *dependent*, say $\delta v = -\delta u$ (recall our assumptions about the δ's). We will follow the latter course (cf. our general remarks in section 3.2.2 below). This ensures that both variables will be of the same order of smallness during the limiting process.

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We first consider a_1: as $\delta u \to 0$, $\delta u < 0$, the arguments to the F_D in eq. 3.15 behave as follows.

$$\frac{e_2 - e_3}{e_2 - 0} \to \frac{1}{2}(3 - \mathrm{i}\sqrt{3})\,, \tag{3.17}$$

$$\frac{e_2 - e_3}{e_2 - \sqrt{\delta u/3}} \to \frac{1}{2}(3 - \mathrm{i}\sqrt{3})\,, \tag{3.18}$$

$$\frac{e_2 - e_3}{e_2 + \sqrt{\delta u/3}} \to \frac{1}{2}(3 - \mathrm{i}\sqrt{3})\,, \tag{3.19}$$

$$\frac{e_2 - e_3}{e_2 - e_1} \to \frac{1}{2}(1 - \mathrm{i}\sqrt{3})\,, \tag{3.20}$$

$$\frac{e_2 - e_3}{e_2 - e_4} \to 0\,, \tag{3.21}$$

$$\frac{e_2 - e_3}{e_2 - e_5} \to 0\,, \tag{3.22}$$

$$\frac{e_2 - e_3}{e_2 - e_6} \to 0\,. \tag{3.23}$$

Furthermore, *the prefactor of the F_D in eq. 3.15 tends to* 0. These limits were obtained using Mathematica. We will now prove that the value of the F_D at the limits just written down is a complex number (i. e., that it is defined for that particular constellation of arguments).

First observe that

$$F_D^{(7)}(\ldots,0,0,0) = F_D^{(4)}(\ldots)\,, \tag{3.24}$$

and

$$F_D^{(4)}\left(\frac{1}{2},-1,-1,-1,\frac{1}{2};1;\frac{1}{2}(3-\mathrm{i}\sqrt{3}),\frac{1}{2}(3-\mathrm{i}\sqrt{3}),\frac{1}{2}(3-\mathrm{i}\sqrt{3}),\frac{1}{2}(1-\mathrm{i}\sqrt{3})\right) = \tag{3.25}$$
$$= F_D^{(2)}\left(\frac{1}{2},-3,\frac{1}{2};1;\frac{1}{2}(3-\mathrm{i}\sqrt{3}),\frac{1}{2}(1-\mathrm{i}\sqrt{3})\right).$$

Since -3 is a negative integer this last F_D reduces to a polynomial in $\frac{1}{2}(3-\mathrm{i}\sqrt{3})$ (see appendix A), the coefficients being rational numbers multiplied by expressions of the form ${}_2F_1(\frac{1}{2}+n,\frac{1}{2};1+n;\frac{1}{2}(1-\mathrm{i}\sqrt{3}))$, where n is some nonnegative integer (${}_2F_1$ is the gaussian hypergeometric function). Since for our constellation of parameters the only singularity of ${}_2F_1$ on the unit circle is at 1, these 'expressions' reduce to (finite) complex numbers. Thus, as promised, the F_D in eq. 3.15 converges at the limits of its arguments.

Therefore (recall that the prefactor tends to 0*), a_1 tends to* 0*, as* $\delta u \to 0$*,* $\delta u < 0$.

We now consider a_D^1: referring to eq. 3.16 one finds that the arguments of the F_D in that equation, taken in the same order as those of the F_D in eq. 3.15, tend to the same limits as indicated in eq. 3.17. Following exactly the same arguments as for a_1 we find that the F_D in the expression for a_D^1 tends to the same (finite) complex number as the one in eq. 3.15. Also, the prefactor in eq. 3.16 tends to 0 as $\delta u \to 0$, $\delta u < 0$.

Therefore, a_D^1 tends to 0 *as* $\delta u \to 0$*,* $\delta u < 0$.

This vanishing of a_1 and a_D^1 reproduces the result stated by Argyres and Douglas [17] in the special limit (recall our restrictions on the δ's) we have considered.

3.2.2 General discussion

The reader might ask why in the previous section we imposed so peculiar restrictions on δu and δv. In the present section, we wish to address precisely this question.

Partly the answer is simple: We imposed these conditions so that we were able to consider the limit of the scalar modes at the Argyres-Douglas point in a straight-forward manner. For instance, we know for sure that it is possible to consider the case $\delta u > 0$, $\delta u = \delta v$, $\delta u \to 0$,[3] and the analysis of this is exactly the same as in the case we have presented in detail (even the arguments of the F_D's tend to the same limits). However, there is a complication involved. Now one has to look at the discriminant of p_1; if it is 0, a degeneration of the 'inner' e's (cf. figure 2) takes place ($e_2 = e_3$, moreover, $e_2 = -\sqrt{-\delta u/3}$), and if it is negative, the homology basis of figure 2 is inadequate for our formulas because then all 'inner' e's are real. One does away with the trouble of the discriminant in this case by setting $\delta u = \delta v$ (this implies positivity of the discriminant). The moral is that it is possible to impose different restrictions on the δ's than we have done in this paper.

In the general case of arbitrary complex $\delta u, \delta v$ with an arbitrary approach to 0 (as in standard calculus) the complications pile up: Firstly, there is the technical problem of keeping track of third roots of complex numbers. This is very difficult, since the expressions involved in the usual Cardano's formula are intrinsically discontinuous due to branch cuts; also, multivaluedness becomes a burden. Secondly, there is the problem that one cannot choose a homology basis once and for all. For instance, if $\delta v > 0$, $\delta u < 0$ the configuration of the 'inner' e's looks like that of those in figure 2, only reflected through the origin. Of course this necessitates the use of a different homology basis than the one shown in figure 2. If the δ's can arbitrarily approach 0, then it is clear that one could start with a homology basis like the one in figure 2 and end up with the need to chose a different one. Our formulas are not suited for such a case.

The question arises: Do a_1,a_D^1 vanish at the Argyres-Douglas point regardless of exactly how the δ's tend to 0? If they should happen to be continuous functions the answer is affirmative: Yes! However, we do not deem their continuity as self-evident, since the e's depend on the δ's in a rather complicated fashion. On some Riemann-surface a_1,a_D^1 most likely *are* continuous. But that seems to be something different.

It is tautological to say that if $a_1(\delta u, \delta v)$,$a_D^1(\delta u, \delta v)$ should happen to be discontinuous at $(0, 0)$, then there would exist some approach of the δ's to 0 which would not yield a vanishing limit.

In light of the particular properties of the Lauricella functions under analytic continuation, we point out the following: The periods a_1 and a_D^1 are given in terms of Lauricella functions $F_D^{(7)}$ of seven arguments. As can be inferred from eqs. 3.15 and 3.16, the arguments depend on the branch points. In our conventions, the three branch points e_1, e_2 and e_3 are all located in a small neighbourhood around 0, if we approach the Argyres-Douglas point. However, generically, they are farther away from 0 than the two other zeroes of the Seiberg-Witten differential $z_{1,2} = \pm\sqrt{u/3}$. It is an easy task to convince oneself, that appropriate choices of δu and δv can invert this situation, such that some

[3]Again, a_1 and a_D^1 vanish.

of the 'small' branch points are closer to zero than the z's. Of course, then the situation is possible where a branch point coincides with one of the z's before the point 0 is approached. Also, it can be arranged that two of the three 'small' branch points coincide first, before the three finally approach zero. All these situations fall in the class of 'singular sub-manifolds' [15].

Let us look specifically at the fourth argument of the periods. In the notation introduced in section 2.2 it reads $x_1 = \frac{e_2-e_3}{e_2-e_1}$ for $a_1(\delta u, \delta v)$ or $x_2 = -\frac{e_3-e_1}{e_2-e_3}$ for $a_D^1(\delta u, \delta v)$, respectively. Clearly, depending on how the three 'small' branch points approach each other and 0, the moduli of the two arguments are related as $|x_1| = 1/|x_2|$ and hence are located around one of the following pairs of singular points: $(1,1)$, $(0,\infty)$, or $(\infty,0)$. Since this fourth argument is associated with a parameter b_x, which is not a negative integer, the corresponding Lauricella function is not simply polynomial in this argument. Thus, in all four cases will we need at least one analytic continuation, either of a_1 or of a_D^1. Due to the specific values of the paramters, each of these analytic continuations will involve a logarithmic divergence of the involved Lauricella function, killed off only by the prefactor.

Next, we look at the last three arguments. Generically, all three approach 0 from three different directions. However, in particular if Λ is allowed to be small and complex, the 'large' branch points might run into each other first, e.g. on the sub-manifold characterized by $\{\delta u, \delta v, \Lambda, w : \delta u = 3w^2, \delta v + 2\Lambda^3 = -2w^3\}$. This is then exactly the difficult situation that the singular point $(x_{1,2}, 0, 0, 0)$ is a common point of various sub-manifolds $x_k = x_l$ for $k, l \in \{4,5,6\}$. Similarly, the first three arguments close in on one common singular point, but this is not really a difficulty, since the corresponding parameters render the Lauricella functions polynomial in their first three variables. Hence, analytic continuation in one of the first three variables is only necessary in the true limit of the Argyres-Douglas point itself, where these variables all tend to infinity. For this, one should note that the zeroes of $\lambda_{\rm SW}$ are generically closer to 0, corresponding to the Argyres-Douglas point, than the 'small' branch points.

Therefore, depending on the choice of the δ's, we may move on or across singular sub-manifolds which will make it necessary to perform the appropriate analytic continuations. In particular, if we have to analytically continue in more than one variable, say x_5 and x_6, great care has to be taken that for an analytic continuation around, say, the point $(1,1)$ we need *two* different continuations, namely for the patches $|x_5| < |x_6|$ and $|x_5| > |x_6|$, respectively. These can be found along the lines set out in the appendix, but they are more complicated than the examples provided there. The discriminant of the curve y^2 tells us where branch points coincide. A special locus of that kind is $\{\delta u, \delta v : 27(\delta v)^2 = 4(\delta u)^3\}$, where two of the three 'small' branch points become equal, say e_2 and e_3. Then $a_1(\delta u, \delta v)$ is well defined around the singular point $(0,0,0,0)$ for its last four variables, while $a_D^1(\delta u, \delta v)$ has to be analytically continued around $(\infty, 0, 0, 0)$ according to A.8. The situation gets more involved, if we allow Λ to vary arbitrarily or if we include massive hypermultiplets. Then, almost any singular sub-manifold can be reached, and the distinction between 'small' and 'large' branch points may cease to exist, leading to higher-dimensional singular sub-manifolds.

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In closing, we would like to emphasize that eqs. 3.15 and 3.16 are, of course, valid without any particular restrictions on the δ's other than that all relevant expression be defined.

4. Conclusion

We have derived general formulas for the scalar modes in Seiberg-Witten theory for gauge group SU(N) in the absence as well as in the presence of massive hypermultiplets. These formulas involve the Lauricella function $F_D^{(n)}$, a multiple hypergeometric function. The advantage of our approach is threefold. Firstly, our formulas can be worked out directly, without the need of first finding the correct system of Picard-Fuchs differential equations and its solutions. Secondly, our formulas work equally well for the case with (massive) hypermultiplets, putting them on equal footing with the ordinary case. Finally, one can immediately extract a well-defined power series expansion for any BPS period from our results, as long as the corresponding homology cycle is the smallest one.[4] Thus, in any given patch of moduli space, the lightest BPS state for this patch can be obtained in a straight-forward manner.

Subsequently, we have applied the formula for the case without hypermultiplets to the Argyres-Douglas point for gauge group SU(3). Specifically, we have considered the limits of the scalar modes corresponding to the cycles α_1,β^1 as the vacuum expectation values u,v tend — in a special way — to those values which define the Argyres-Douglas point. In accord with the result of Argyres and Douglas we have found that the limit of both these modes is indeed zero.

As already pointed out, this raises the question of what happens when u,v *arbitrarily* tend to the prescribed values. We cannot tell for sure. To begin with, we think it is conceivable that there exists some mode of approach which yields a limit other than 0. This possibility is intimately connected with — indeed, is equivalent to — the question whether the scalar modes are continuous functions of the vacuum expectation values. With the existence of phase transitions in Seiberg-Witten theory in mind, we believe that it is necessary to prove the continuity of the scalar modes rather than just assume it.

Even if the scalar modes should turn out to be continuous it is quite puzzling to observe that in the special case we have considered the Lauricella functions involved in our formulas tend to the same values, so that the vanishing of the scalar modes is determined by the vanishing of the prefactors. What is puzzling about this is that it looks as if a_1,a_D^1 become linearly dependent at the Argyres-Douglas point.

Surely more work is needed in order to fully understand the analytic properties of the Lauricella system $F_D^{(n)}$. Furthermore, we have a price to pay for being able to compute periods of the Seiberg-Witten form explicitly: We have to change the variables from the vev's of the scalar field to quotients of differences of branch points. This is a highly non-linear and multi-valued map. However, we would like to point out that the alternative, where one works with the vev's u_k, means to find the system of Picard-Fuchs equations and

[4] This is a sufficient condition. Of course, easily computable power series expansions exist for many more cases.

to solve it. This has been achieved in the case of gauge group SU(3), [4, 26], but the results are given in terms of the Appell function F_4 of two arguments (which are relatively simple functions of u_1 and u_2), for which no nice integral representation is available. Furthermore, the problem of finding analytic continuations is similarly complicated. There are regions in moduli space where simple monodromy transformations will yield the desired results, but a complete set of analytic continuations cannot be obtained in this way. In addition, the Appell function F_4 and its generalization to more than two variables has an even worse structure under analytic continuation than the Lauricella $F_D^{(n)}$ system. Therefore, we believe that our approach offers several advantages when we wish to investigate Seiberg-Witten theory in all of moduli space.

First of all, the Lauricella system $F_D^{(n)}$ together with the related functions $D_{p,q}^{(n)}$ are the simplest of all the generalized hypergeometric systems of several variables. It might be tedious, but it is possible to compute analytic continuations for them such that one can access all of moduli space, not only the semi-classical regime. Moreover, the formulas are explicit and thus allow direct computations of the scalar modes.

For all the attractiveness of our approach we lose the direct relation of the vev's to the scalar field. It depends on what one wishes to consider as 'fundamental.' From the point of view of effective field theory, the vev's are the physically given parameters, and the Riemann surface is an auxiliary construction. From the vista of string theory, the Riemann surface has a certain reality.

Acknowledgments

N.A. thanks Christian Sämann for useful discussions. M.F. thanks Albrecht Klemm and Werner Nahm for useful discussions. The work of M.F. is supported by the European Union network HPRN-CT-2002-00325 (EUCLID) and in part by the string theory network (SPP no. 1096), Fl 259/2-2, of the Deutsche Forschungsgemeinschaft.

A. Lauricella $F_D^{(n)}$

The purpose of this appendix is to collect various information pertaining to Lauricella $F_D^{(n)}$. Some of the formulas have already made their appearance in one or the other section of the main text. The major reference for Lauricella $F_D^{(n)}$ and other 'multiple hypergeometric functions' is [15], from which we will cite freely.

A.1 The definition

Lauricella $F_D^{(n)}$ is a function of n complex variables and $n+2$ parameters, defined by the power series

$$F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n} m_1! \cdots m_n!} x_1^{m_1} \cdots x_n^{m_n}\,, \tag{A.1}$$

whenever $|x_1|, \ldots, |x_n| < 1$ and by analytic continuation elsewhere. The symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the so-called Pochhammer symbol.

The function $F_D^{(n)}$ has the integral representation

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}\prod_{i=1}^n(1-tx_i)^{-b_i}\,\mathrm{d}t = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\,F_D^{(n)}(a,b_1,\dots,b_n;c;x_1,\dots,x_n)\,, \tag{A.2}$$

if $\mathrm{Re}(a)$ and $\mathrm{Re}(c-a)$ are positive. This is proved using the binomial theorem. See [16], appendix B for a detailed derivation.

A number of facts can be read off the power series representation A.1:

1. If one of the variables of $F_D^{(n)}$, say x_i, is equal to 0, then the $F_D^{(n)}$ reduces to a $F_D^{(n-1)}$:

$$\begin{aligned} &F_D^{(n)}(a,b_1,\dots,b_i,\dots,b_n;c;x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_n) = \\ &\quad = F_D^{(n-1)}(a,b_1,\dots,\widehat{b_i},\dots,b_n;c;x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)\,, \end{aligned} \tag{A.3}$$

   where $\widehat{\ }$ denotes omission.

2. If two variables have equal values, say $x_i = x_j$, a similar reduction takes place:

$$\begin{aligned} &F_D^{(n)}(a,b_1,\dots,b_i,\dots,b_j,\dots,b_n;c;x_1,\dots,x_i,\dots,x_j,\dots,x_n) = \\ &\quad = F_D^{(n-1)}(a,b_1,\dots,b_{i-1},b_i+b_j,b_{i+1},\dots,\widehat{b_j},\dots,b_n;c;x_1,\dots,x_i,\dots,\widehat{x_j},\dots,x_n)\,. \end{aligned} \tag{A.4}$$

3. If b_i is a negative integer, then the part of the series corresponding to x_i terminates after a finite number of terms (because $(b_i)_n = 0$ for $n > |b_i|$) and thus reduces to a polynomial in x_i. In this case, the modulus of x_i is immaterial for the validity of A.1.

If $|x_i| \geq 1$, for some $i \in \{1,\dots,n\}$, then the power series representation A.1 is not valid anymore. Rather, one must have recourse to an analytic continuation of $F_D^{(n)}$. This is effected by writing

$$\begin{aligned} &F_D^{(n)}(a,b_1,\dots,b_n;c;x_1,\dots,x_n) = \\ &\quad = \sum_{m_1=0}^{\infty}\cdots\sum_{m_{i-1}=0}^{\infty}\sum_{m_{i+1}=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\dots+m_{i-1}+m_{i+1}+\dots+m_n}\prod_{\ell\neq i}(b_\ell)_{m_\ell}}{(c)_{m_1+\dots+m_{i-1}+m_{i+1}+\dots+m_n}\prod_{\ell\neq i}m_\ell!}\times \\ &\quad\quad \times\prod_{\ell\neq i}x_\ell^{m_\ell}\,{}_2F_1\left(a+\sum_{\ell\neq i}m_\ell, b_i; c+\sum_{\ell\neq i}m_\ell; x_i\right)\,, \end{aligned} \tag{A.5}$$

and employing a suitable continuation formula for the gaussian hypergeometric function ${}_2F_1$. Erdélyi et al. [27] as well as Becken and Schmelcher [28] give several such continuation formulas. Especially the latter reference appears to be quite exhaustive. Also, see the next section A.2.

Nevertheless, sometimes one can get along without any explicit continuation formula, just as we have done in section 3.2.

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A.2 Analytic continuation of Lauricella functions

The Seiberg-Witten periods are analytic functions 'everywhere' in the moduli space, i.e. for generic values of either the vacuum expectation values u_k or the branch points e_ℓ. However, it is clear that this is not necessarily the case in the singular regions where one or more branch points e_ℓ become identical. In fact, a typical feature of a dual pair (a, a_D) of Seiberg-Witten periods corresponding to a dual pair of homology cycles (α, β) is that with a vanishing cycle $\alpha \to 0$, only $a(u)$ becomes small, while the dual period $a_D(u)$ diverges logarithmically.

Precisely this is it what makes the Argyres-Douglas point so interesting: a pair of dual periods *simultaneously* become small, i.e. two particles dual to each other under some sort of electromagnetic duality, simultaneously become massless. As a consequence of this the theory is conformally invariant in this regime.

There is another interesting fact about such points in moduli space, where intersecting homology cycles vanish at the same time. If one uses the branch points as the natural coordinates to parametrize the theory, it is known from the theory of generalized hypergeometric functions that a complete set of analytic continuations cannot be given entirely in terms of the same class of functions one starts out with (cf. [15]). This is to be contrasted with the well-known result that the ordinary gaussian hypergeometric function admits analytic continuations everywhere in the complex plane, which again can be expressed as linear combinations of gaussian hypergeometric functions (cf., e.g. [28]). This can be used, for example, to find analytic continuation formulas for the Lauricella $F_D^{(n)}$ function, as long as we only need to continue one of its arguments outside the unit circle of convergence. As soon as we wish to have more than one argument outside the unit circle, things become complicated. In case of the Lauricella $F_D^{(n)}$ function, one is confronted with the following problem:

Within its region of convergence, the Lauricella $F_D^{(n)}$ series can be represented in the form of an Euler-type integral, i.e. an integral along a simple loop, which we choose to be one of the homology cycles. More generally, Pochhammer double loops might be admitted as well. The point is, that only two of the singular points of the differential are enclosed by the loop. The thus defined functions possess analytic continuations which, for generic values of the parameters, can again be given in terms of multi-variable power series (perhaps multiplied by a common fractional power). However, such analytic continuations are only valid outside the unit ball within cones delimited by the singular hyperplanes, given by coinciding singular points. Thus, one needs a considerably larger set of analytic continuations than in the single-variable case. Morevoer, not all of these analytic continuations can be represented by Euler-type integrals. Exton [15] mentions that at least so-called three-foil loops are necessary to be able to represent a full set of analytic continuations in terms of integrals. Three-foil loops are three times self-intersecting loops which enclose three different sets of singular points.

The linearity of the integral guarantees that, in principle, we could eliminate the internal dependence on the zeros of the Seiberg-Witten differential by decomposing the integrand. The resulting Lauricella functions of fewer arguments depend solely on the branch points via the quotients of differences of them, as introduced earlier. However, generic small variations of a subset of branch points which are in a small neighborhood around a physically interesting singular point such as the Argyres-Douglas point, have to be treated carefully. On the one hand, we might need several analytic continuations, because the small variation crosses a boundary of the cone of convergence, which coincides with the situation that the initial choice of a homology basis ceases to be valid. On the other hand, we even might run into a region where no Euler-type integral representation exists.

In order to study the analytic continuations of the Lauricella functions of type F_D, one needs a further class of related functions, defined by the expansions

$$D^{(n)}_{p,q}(a,b_1,\ldots,b_n;c,c';x_1,\ldots,x_n) = \tag{A.6}$$
$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_{p+1}+\cdots+m_n-m1-\cdots-m_p}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_{q+1}+\cdots+m_n-m_1-\cdots-m_p}c'_{m_{p+1}+\cdots+m_q}m_1!\cdots m_n!}x_1^{m_1}\cdots x_n^{m_n}\,,$$

where $0\le p\le q\le n$. It is important to note that these functions, which appear in the analytic continuations of the Lauricella $F_D^{(n)}$ functions, do not possess Euler-type integral representations. The simplest known integral representation is in fact a Pochhammer double loop integral involving a Lauricella function in its kernel, namely

$$\begin{aligned}
&\frac{(2\pi\mathrm{i})^2}{\Gamma(a)\Gamma(a')\Gamma(2-a-a')}D^{(n)}_{p,q}(a+a'-1,b1,\ldots,b_n;c,c';x_1,\ldots,x_n) = \\
&\quad= \int \mathrm{d}u\,(-u)^{-a'}(u-1)^{-a}F_D^{q-p}\left(a',b_{p+1},\ldots,b_q;c';\frac{x_{p+1}}{u},\ldots,\frac{x_q}{u}\right)\times \\
&\quad\times D^{(p)}_{n-q+p,n-q+p}\left(a,b_{q+1},\ldots,b_n,b_1,\ldots,b_p;c,c;\frac{x_{q+1}}{1-u},\right. \\
&\qquad\qquad\left.\ldots,\frac{x_n}{1-u},\frac{x_1}{1-u},\ldots,\frac{x_p}{1-u}\right),
\end{aligned}\tag{A.7}$$

where the Pochhammer double loop encircles 0 and 1. We will now give three cases of analytic continuations. Other cases can be obtained in a similar way. One starts with the still simple case that only one argument is either close to 1 or ∞. This case can be developed along the lines set out in [15] by rewriting the multiple series in such a way that the innermost summations is replaced by ordinary gaussian hypergeometric series, for which the analytic continuation is known, see eq. A.5 above. This yields the following results: For the region near infinity, i.e. $1/|x_n|<1$, the analytic continuation reads

$$\begin{aligned}
F_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n) &= \frac{\Gamma(c)\Gamma(b_n-a)}{\Gamma(b_n)\Gamma(c-a)}(-x_n)^{-a}\times \\
&\times F_D^{(n)}\left(a,b_1,\ldots,b_{n-1},1-c+a;1-b_n+a;\frac{x_1}{x_n},\right. \\
&\qquad\left.\ldots,\frac{x_{n-1}}{x_n},\frac{1}{x_n}\right)+\frac{\Gamma(c)\Gamma(a-b_n}{\Gamma(a)\Gamma(c-b_n)}(-x_n)^{-b_n}\times \\
&\times D^{(n)}_{1,1}\left(a-b_n,b_n,b_1,\ldots,b_{n-1};c-b_n,c-b_n;\frac{1}{x_n},\right.
\end{aligned}$$

$$x_1, \ldots, x_{n-1} \Big) \,. \tag{A.8}$$

If $|1 - x_n| < 1$, we are in the region close to 1, and the analytic continuation now reads

$$\begin{aligned}
&F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \\
&\quad = \frac{\Gamma(c)\Gamma(c-a-b_n)}{\Gamma(c-a)\Gamma(c-b_n)} D_{0,n-1}^{(n)}(a, b_1, \ldots, b_n; a+b_n-c+1, c-b_n; x_1, \\
&\qquad\qquad \ldots, x_{n-1}, 1-x_n) + \\
&\quad + \frac{\Gamma(c)\Gamma(a+b_n-c)}{\Gamma(a)\Gamma(b_n)} (1-x_n)^{c-a-b_n} \times \\
&\quad \times D_{0,n-1}^{(n)}(c-b_n, b_1, \ldots, b_{n-1}, c-a; c-a-b_n+1, c-b_n; x_1, \ldots, x_{n-1}, 1-x_n) \,.
\end{aligned} \tag{A.9}$$

The first of the two continuations has the advantage that, in the region of large $|x_n|$, the first term on the right hand side is convergent even if all the other x_i, $i \neq n$, are close to 1. Thus, to find the analytic continuation in the case that one argument is large and another is close to 1, one only need to seek the analytic continuation of the second term on the right hand side. We will give here the result for the case that $|x_1|$ is large and $|x_n|$ is close to 1, since other cases can easily be obtained by permutations.

$$\begin{aligned}
&F_D^{(n)}(a, b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \\
&\quad = \frac{\Gamma(c)\Gamma(b_1-a)}{\Gamma(b_1)\Gamma(c-a)} (-x_1)^{-a} F_D^{(n)}\left(a, 1-c+a, b_2, \ldots, b_n; 1-b_1+a; \frac{1}{x_1}, \frac{x_2}{x_1}, \ldots \frac{x_n}{x_1}\right) + \\
&\quad + \frac{\Gamma(c)}{\Gamma(a)} (-x_1)^{-b_1} \left(\frac{\Gamma(c-a-b_n)}{\Gamma(c-b_1-b_n)} D_{1,2}^{(n)}(a-b_1, b_1, b_n, b_2, \ldots, b_{n-1}; c-b_1-b_n, \right. \\
&\qquad\qquad a+b_n-c+1; \frac{1}{x_1}, 1-x_n, x_2, \ldots, x_{n-1}) + \\
&\qquad + \frac{\Gamma(a+b_n-c)}{\Gamma(b_n)} (1-x_n)^{c-a-b_n} \times \\
&\qquad \times D_{1,2}^{(n)}(c-b_1-b_n, b_1, c-a, b_2, \ldots, b_{n-1}; \\
&\qquad\qquad \left. c-b_1-b_n, c-a-b_n+1; \frac{1}{x_1}, 1-x_n, x_2, \ldots, x_{n-1}) \right) \,.
\end{aligned} \tag{A.10}$$

Unfortunately, these formulas are valid only for generic values of the parameters. In the cases relevant for the Seiberg-Witten periods, we have certain relations such as $a = b_i$ for some i which will cause singularities, if one attempts to analytically continue in the coordinate x_i. To obtain the correct answer, one has to take one further step, namely a limiting procedure. This is the well known Frobenius process, which essentially is nothing else than to consider the limit of $b_i = a + \epsilon$ for $\epsilon \to 0$. As an example, we present here one particular instance, namely

$$\begin{aligned}
&F_D^{(n)}(a, b_1, \ldots, b_{n-1}, a; c; x_1, \ldots, x_n) = \\
&\quad = \Gamma\begin{bmatrix} c \\ a, c-a \end{bmatrix} (-x_n)^{-a} \sum_M \sum_{m_n=0}^{\infty} \Gamma\begin{bmatrix} c-a-|M| \\ c-a+|M| \end{bmatrix} \frac{(a)_{|M|+m_n}(1-c+a)_{2|M|+m_n}}{(|M|+m_n)!m_n!} \times
\end{aligned}$$

$$\times \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} \left(\log(-x_n) + h_{m_n}\right) \left(\frac{x_1}{x_n}\right)^{m_1} \cdots \left(\frac{x_{n-1}}{x_n}\right)^{m_n - 1} \left(\frac{1}{x_n}\right)^{m_n} + \tag{A.11}$$

$$+\Gamma\begin{bmatrix} c, c-a \\ a \end{bmatrix} (-x_n)^{-a} \sum_M \sum_{m_n=0}^{|M|-1} \frac{(a)_{m_n}\Gamma(|M|-m_n)}{m_n!(c-a)_{|M|-m_n}} \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \left(\frac{1}{x_n}\right)^{m_n},$$

where

$$h_{m_n} = \psi(1+|M|+m_n) + \psi(1+m_n) - \psi(a+|M|+m_n) - \psi(c-a-m_n)\,, \tag{A.12}$$

and we have made use of multindex notation, so that $M = (m_1, \ldots, m_{n-1})$, $|M| = \sum_{i=1}^{m_n} m_i$ and summation over M means summation over each m_i $(i = 1, \ldots, n-1)$ from 0 to ∞.

This result can also be obtained 'directly' by using a suitable continuation formula for the gaussian hypergeometric function.

Since such limiting procedures make the formulas extremely cumbersome, it is easier to work with the generic formulas, perform the necessary expansions with a computer algebra package and to then take the limit. The Lauricella functions which we encountered in our study of the SU(2) case (cf. section 3.1) can be written in the following form: For $a(u)$, one has $F_D^{(3)}(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, \frac{x}{2-x}, \frac{x}{2})$, where $x = 1 - \frac{e_1}{e_4}$ with the notations used there, i.e. $e_1 = \sqrt{u - \Lambda^2}$ and $e_4 = \sqrt{u + \Lambda^2}$. For small x, this has a good power series expansion, namely,

$$F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, \frac{x}{2-x}, \frac{x}{2}\right) = 1 - \frac{3}{4}x + \frac{5}{32}x^2 + \frac{3}{123}x^3 - \frac{169}{262144}x^4 - \frac{1131}{1048576}x^5 + \cdots. \tag{A.13}$$

Now, again for small x, the dual period is proportional to $F_D^{(3)}(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; 2, 2\frac{x-1}{x}, 2\frac{x-1}{x-2})$ which calls for an analytic continuation valid near the point $(0, \infty, 1)$. Actually, the first argument has modulus greater than 1, but since the Lauricella function is only polynomial in its first argument, we do not have to perform an analytic continuation for it. We find thus for the dual period

$$\begin{aligned} F_D^{(3)}&\left(\frac{1}{2}, \frac{1}{2}, -2, \frac{1}{2}; 1; 2\frac{x-1}{x}, 2, 2\frac{x-1}{x-2}\right) = \\ &= -\frac{\sqrt{2x}}{\pi}\left(2 + 3x + \frac{27}{8}x^2 + \frac{19}{6}x^3 + \frac{9559}{4096}x^4 + \frac{12019}{12288}x^5 + \cdots + \right. \\ &\qquad + \log(2)\left(3 + \frac{15}{4}x + \frac{135}{32}x^2 + \frac{561}{128}x^3 + \frac{989}{256}x^4 + \frac{11169}{4096}x^5 + \ldots\right) - \\ &\qquad \left. - \log(x)\left(1 + \frac{5}{4}x + \frac{41}{32}x^2 + \frac{147}{128}x^3 + \frac{193}{256}x^4 + \frac{575}{4096}x^5 + \ldots\right)\right). \end{aligned} \tag{A.14}$$

Of course, we can also obtain $a_D(u)$ from $a(u)$ by a suitable analytic continuation. For this, one notes that the Lauricella functions are related to each other by (for the moment ignoring the fact that the three variables are all functions of $x = 1 - \frac{e_1}{e_4}$)

$$a(u) \propto F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; x, y, z\right),$$

$$a_D(u) \propto F_D^{(3)}\left(\frac{1}{2}, -2, \frac{1}{2}, \frac{1}{2}; 1; \frac{x}{z}, 1-\frac{1}{y}, 1-y\right), \tag{A.15}$$

where the prefactors have been omitted. Such relations can be used, for example, to find the curve of marginal stability, where $a(u)/a_D(u)$ is real.

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